International Journal of Trends in Mathematics Education Research
Vol. 7, No. 3 (2024), pp. 141~146
ISSN 2621-8488 (online)
DOI: https://doi.org/10.33122/ijtmer.v7i3.324

SAINTIS publishing

Research Article/Review Article

Isomorphisms on Strong Fuzzy Graphs

Yuliana Trisanti[1] & Hairus Saleh[2*]

[1] Department of Management, Universitas Madura, Pamekasan, Indonesia 69371
[2] Department of Mathematics Education, Universitas Madura, Pamekasan, Indonesia 69371
*Corresponding Author: hairuss_math@unira.ac.id | Phone: +6285284000556

Received: 10 August 2024 *Revised: 24 August 2024* *Accepted: 27 August 2024* *Available online: 10 September 2024*

ABSTRACT

Fuzzy graph is a graph consisting of pairs of vertices and edges that have a membership degree that contains a closed interval of real numbers $[0,1]$ on each edge and vertex. A fuzzy graph $G(\sigma,\mu)$ is said to be a strong fuzzy graph if the membership degree of edge uv with the minimum membership degree of vertex u and the same membership degree of vertex v for each edge uv with $uv \in S$. This paper explains the properties of fuzzy graph isomorphism including weak isomorphism, co-weak isomorphism, and isomorphism in strong fuzzy graphs.

Keywords: Strong Fuzzy Graphs; Isomorphism; Strength of Connectedness

1. INTRODUCTION

Fuzzy Graph is an extension of the Rigid Graph. The first definition of a fuzzy graph was introduced by Kaufmann in 1973 based on the Zadeh fuzzy relation. A more detailed definition was given by A. Rosenfeld in 1975, considering fuzzy relations on fuzzy sets and building a theory of fuzzy graphs. Rosenfeld has produced several concepts such as bridges, paths, cycles and fuzzy trees and determined their properties. At the same time, Yoh and Bang also introduced the concept of connectedness in fuzzy graphs. This paper will discuss the properties of fuzzy graph isomorphism, including weak isomorphism, co-weak isomorphism, and isomorphism in strong fuzzy graphs developed by A. Nagoor Gani and J. Malarvizhi (2009).

2. STRONG FUZZY GRAPH

The following will provide definitions of sharp graphs, fuzzy graphs, connectedness in fuzzy graphs and strong fuzzy graphs.

Definition 1 Let S be a set of vertices. A fuzzy graph $G(\sigma,\mu)$ is a pair of functions where:

$\sigma : S \to [0,1]$

$\mu : S \times S \to [0,1]$

such that $\mu(uv) \le \sigma(u) \wedge \sigma(v), \forall u, v \in S$ where σ is the vertex membership degree and μ is the edge membership degree of the fuzzy graph. The notation $\wedge = inf\{\sigma(u), \sigma(v)\}, \forall u, v \in S$.

Definition 2 Vertex u and v are said to be adjacent if there is an edge e connecting the two vertex or $e = uv$. While u and v are said to be incident with edge e and e is said to be incident with u and v.

Example 1 Given a set of vertex S, namely $S = \{a,b,c,d,e\}$ and a fuzzy graph $G(\sigma,\mu)$ where the membership degrees of the vertex are $\sigma(a) = 0{,}6, \sigma(b) = 0{,}4, \sigma(c) = 0{,}6, \sigma(d) = 0{,}4, \sigma(e) = 0{,}5$ and the membership degrees of the edges are $\mu(ab) = 0{,}3, \mu(bc) = 0{,}2,$ $\mu(be) = 0{,}1, \mu(cd) = 0{,}2, \mu(ce) = 0{,}4, \mu(de) = 0{,}3$, then the fuzzy graph $G(\sigma,\mu)$ is as follows

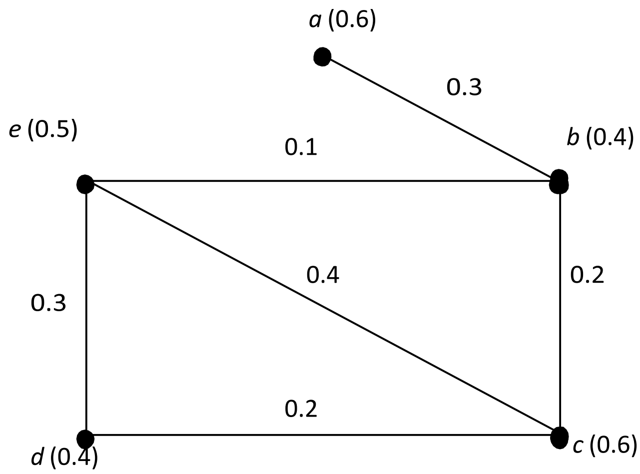

Figure 1. Fuzzy Grah G

Definition 3 A path in a fuzzy graph $G(\sigma,\mu)$ is a sequence of distinct edges $u_0u_1, u_1u_2, u_2u_3, \dots, u_{k-1}u_k$ such that $\mu(u_{i-1}u_i) > 0$ and $1 \le i \le k$. Here k is called the length of a path. A consecutive pair $u_{i-1}u_i$ is called an arc on the path. The path connecting a vertex u to v is denoted by $P(u - v)$.

Definition 4 A fuzzy graph $G(\sigma,\mu)$ is called a strong fuzzy graph if $\mu(uv) \le \sigma(u) \wedge \sigma(v), \forall(uv) \in \mu^*$.

Definition 5 If u and v are vertex in $G(\sigma,\mu)$ with paths $u = v_0, v_1, v_2, \dots, v_{k-1}, v_k$ and if u and v are connected by a path, then the strength of the path is defined as $\wedge_{i=1}^{k} v_{i-1}v_i$ which is the strength of the weakest arc.

Example 2 Given a fuzzy graph $G(\sigma,\mu)$ in Figure 1, the path strength from vertex a to vertex c will be sought, which is defined as the strength of the weakest arc, then obtained:

Path strength from point a to point c

Path strength $abc = inf\{0.3,0.2\} = 0.2$

Path strength $abec = inf\{0.3,0.1,0.4\} = 0.1$

Path strength $abedc = inf\{0.3,0.1,0.3,0.2\} = 0.1$

Definition 6 If u and v are connected by a path of length k then $\mu^k(uv)$ is defined $\mu^k(uv) = Sup\{\mu(uv_1), \mu(v_1v_2), \dots, \mu(v_{k-1}v)\}, \forall u, v_1, \dots, v_{k-1} \in S$.

Example 3 Given a fuzzy graph $G(\sigma,\mu)$ in Figure 1, the path strength from vertex a to vertex c with length k is obtained as follows:

Path strength $abc = sup\{0.3,0.2\} = 0.3$

Path strength $abec = sup\{0.3,0.1,0.4\} = 0.4$

Path strength $abedc = sup\{0.3,0.1,0.3,0.2\} = 0.3$

Definition 7 If $u, v \in S$ the strength of connectedness between u and v is defined as $\mu^\infty(uv) = Sup\{\mu^k(uv), k = 1,2, \dots\}$.

Example 4 Continuing Example 3, the strength of the connection from vertex a to vertex c is obtained: $\mu^\infty(ac) = Sup\{0.2,0.1,0.1\} = 0.2$.

Definition 8 A uv edge is called a strong edge if $\mu(uv) \geq \mu^{\infty}(uv)$.

Definition 9 A fuzzy graph $G(\sigma, \mu)$ is a strong fuzzy graph if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for every $uv \in \mu^*$.

Example 5 Given a fuzzy graph $G_1(\sigma_1, \mu_1)$ with a vertex set S_1 namely $S_1 = \{a, b, c, d, e\}$ where $\sigma_1: S_1 \longrightarrow [0,1]$ and $\mu_1: S_1 \times S_1 \longrightarrow [0,1]$ are defined as follows:

a. $\sigma_1(a) = 0.5, \sigma_1(b) = 0.3, \sigma_1(c) = 0.2, \sigma_1(d) = 0.6, \sigma_1(e) = 0.4$.

b. $\mu_1(ae) = 0.4, \mu_1(bc) = 0.2, \mu_1(bd) = 0.3, \mu_1(be) = 0.3, \mu_1(cd) = 0.2, \mu_1(de) = 0.4$.

It will be shown that the fuzzy graph $G_1(\sigma_1, \mu_1)$ is a strong fuzzy graph.

1. $\mu_1(ae) = \sigma_1(a) \wedge \sigma_1(e) = 0.5 \wedge 0.4 = 0.4$
2. $\mu_1(bc) = \sigma_1(b) \wedge \sigma_1(c) = 0.3 \wedge 0.2 = 0.2$
3. $\mu_1(bd) = \sigma_1(b) \wedge \sigma_1(d) = 0.3 \wedge 0.6 = 0.3$
4. $\mu_1(be) = \sigma_1(b) \wedge \sigma_1(e) = 0.3 \wedge 0.4 = 0.3$
5. $\mu_1(cd) = \sigma_1(c) \wedge \sigma_1(d) = 0.2 \wedge 0.6 = 0.2$
6. $\mu_1(de) = \sigma_1(d) \wedge \sigma_1(e) = 0.6 \wedge 0.4 = 0.4$

So, the fuzzy graph $G_1(\sigma_1, \mu_1)$ is a strong fuzzy graph because $\mu_1(uv) = \sigma_1(u) \wedge \sigma_1(v)$ for every $uv \in {\mu_1}^*$.

3. ISOMORPHISM IN STRONG FUZZY GRAPHS

Definition 10 Given fuzzy graphs $G(\sigma, \mu)$ and $G'(\sigma', \mu')$ with vertex sets S and S' respectively. A mapping $h: S \to S'$ with h bijective homomorphism is called a weak isomorphism if it satisfies $\sigma(u) = \sigma'(h(u))$ for every $u \in S$. Furthermore, the fuzzy graph $G(\sigma, \mu)$ is said to be weak isomorphic to $G'(\sigma', \mu')$.

Definition 11 Given fuzzy graphs $G(\sigma, \mu)$ and $G'(\sigma', \mu')$ with vertex sets S and S' respectively. A mapping $h: S \to S'$ with h a bijective homomorphism is called a co-weak isomorphism if it satisfies $\mu(uv) = \mu(h(u)h(v))$ for every $u, v \in S$. Furthermore, the fuzzy graph $G(\sigma, \mu)$ is said to be co-weak isomorphic to $G'(\sigma', \mu')$.

Definition 12 Given fuzzy graphs $G(\sigma, \mu)$ and $G'(\sigma', \mu')$ with vertex sets S and S' respectively. A mapping $h: S \to S'$ with h bijective homomorphism is called isomorphism if it satisfies:

1) $\sigma(u) = \sigma'(h(u))$ for every $u \in S$
2) $\mu(uv) = \mu(h(u)h(v))$ for every $u, v \in S$

Furthermore, the fuzzy graph $G(\sigma, \mu)$ is said to be isomorphic to $G'(\sigma', \mu')$ and is denoted by $G \cong G'$.

Theorem 1 Let $G \cong G'$, G is a strong fuzzy graph if and only if G' is also a strong fuzzy graph.

Proof ($\rightarrow$) Suppose $G \cong G'$ and $G(\sigma, \mu)$ are strong fuzzy graphs. We will show that $G'(\sigma', \mu')$ is also a strong fuzzy graph. Since $G \cong G'$than $\sigma(u) = \sigma'(h(u))$ and $\mu(uv) = \mu(h(u)h(v))$ for every $u, v \in S$, so that fuzzy graph G is a strong fuzzy graph, than $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for every $uv \in \mu^*$. So $G'(\sigma', \mu')$ is also a strong fuzzy graph.

($\leftarrow$) Let $G \cong G'$ and $G'(\sigma', \mu')$ is a strong fuzzy graph, it will be shown that $G(\sigma, \mu)$ is also strong fuzzy graph.

$G'(\sigma', \mu')$ is a strong fuzzy graph, than $\mu'(h(u)h(v)) = \sigma'(h(u)) \wedge \sigma'(h(v))$ for every $uv \in \mu^*$. So that $\mu(uv) = \sigma'(h(u)) \wedge \sigma'(h(v))$, $\mu(uv) = \sigma(u) \wedge \sigma(v)$. So $G(\sigma, \mu)$ is also strong fuzzy graphs.

Example 6 Given a fuzzy graph $G(\sigma,\mu)$ and a fuzzy graph $G'(\sigma',\mu')$ as follows:

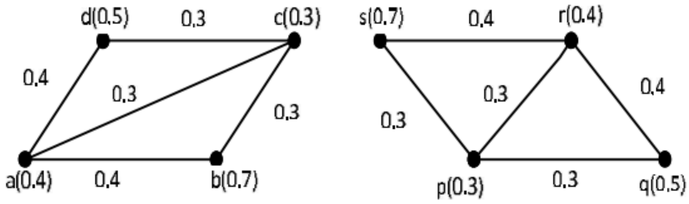

Figure 2. Fuzzy Grah G and G'

with the mapping $h: S \to S'$ which is a bijective mapping defined $h(a) = r, h(b) = s, h(c) = p, h(d) = q$.

We will show that h satisfies the isomorphism properties:

$\sigma(a) = 0.4$ dan $\sigma'(h(a)) = \sigma'(r) = 0.4$

$\sigma(b) = 0.7$ dan $\sigma'(h(b)) = \sigma'(s) = 0.7$

$\sigma(c) = 0.3$ dan $\sigma'(h(c)) = \sigma'(p) = 0.3$

$\sigma(d) = 0.5$ dan $\sigma'(h(d)) = \sigma'(q) = 0.5$

$\mu(ab) = 0.4$ dan $\mu'(h(a)h(b)) = \mu'(rs) = 0.4$

$\mu(bc) = 0.3$ dan $\mu'(h(b)h(c)) = \mu'(sp) = 0.3$

$\mu(cd) = 0.3$ dan $\mu'(h(c)h(d)) = \mu'(pq) = 0.3$

$\mu(ad) = 0.4$ dan $\mu'(h(a)h(d)) = \mu'(rq) = 0.4$

$\mu(ac) = 0.3$ dan $\mu'(h(a)h(c)) = \mu'(rp) = 0.3$

So, it can be concluded that h is a bijective mapping that satisfies the isomorphism properties such that $G(\sigma,\mu)$ is isomorphic to $G'(\sigma',\mu')$.

It will be shown whether $G(\sigma,\mu)$ is a strong fuzzy graph.

1) $\mu(cd) = \sigma(c) \wedge \sigma(d) = 0.3 \wedge 0.5 = 0.3$
2) $\mu(ac) = \sigma(a) \wedge \sigma(c) = 0.4 \wedge 0.3 = 0.3$
3) $\mu(bc) = \sigma(b) \wedge \sigma(c) = 0.7 \wedge 0.3 = 0.3$
4) $\mu(ad) = \sigma(a) \wedge \sigma(d) = 0.4 \wedge 0.5 = 0.4$
5) $\mu(ab) = \sigma(a) \wedge \sigma(b) = 0.4 \wedge 0.7 = 0.4$

so, $G(\sigma,\mu)$ strong fuzzy graph.

It will also be shown whether $G'(\sigma',\mu')$ is a strong fuzzy graph.

1) $\mu'(pq) = \sigma'(p) \wedge \sigma'(q) = 0.3 \wedge 0.5 = 0.3$
2) $\mu'(pr) = \sigma'(p) \wedge \sigma'(r) = 0.3 \wedge 0.4 = 0.3$
3) $\mu'(ps) = \sigma'(p) \wedge \sigma'(s) = 0.3 \wedge 0.7 = 0.3$
4) $\mu'(qr) = \sigma'(q) \wedge \sigma'(r) = 0.5 \wedge 0.4 = 0.4$
5) $\mu'(rs) = \sigma'(r) \wedge \sigma'(s) = 0.4 \wedge 0.7 = 0.4$

So, $G'(\sigma',\mu')$ is also a strong fuzzy graph.

Thus $G(\sigma,\mu)$ is a strong fuzzy graph if and only if $G'(\sigma',\mu')$ is also a strong fuzzy graph.

Theorem 2 Given that $G(\sigma,\mu)$ is co-weak isomorphic to $G'(\sigma',\mu')$, if $G'(\sigma',\mu')$ is a strong fuzzy graph then $G(\sigma,\mu)$ is also a strong fuzzy graph.

Proof Suppose $G(\sigma,\mu)$ is co-weak isomorphic to $G'(\sigma',\mu')$ and $G'(\sigma',\mu')$ is a strong fuzzy graph. It will be proven that $G(\sigma,\mu)$ is also a strong fuzzy graph.

The fuzzy graph $G(\sigma,\mu)$ isomorphically co-weak with $G'(\sigma',\mu')$ then $\sigma(u) \leq \sigma'(h(u))$ for every $u \in S$ and $\mu(uv) = \mu'(h(u)h(v))$ for every $u, v \in S$.

The fuzzy graph $G'(\sigma',\mu')$ is a strong fuzzy graph, then $\mu'(h(u)h(v)) = \sigma'(h(u)) \wedge \sigma'(h(v))$ for every $uv \in \mu^*$. From the facts above, it can be concluded that

$$\mu'(h(u)h(v)) = \sigma'(h(u)) \wedge \sigma'(h(v))$$

$$\mu(uv) = \sigma'(h(u)) \wedge \sigma'(h(v))$$

$\mu(uv) = \sigma(u) \wedge \sigma(v)$ for every $uv \in \mu^*$.

So, $G(\sigma,\mu)$ is also a strong fuzzy graph because $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for every $uv \in \mu^*$.

Corollary Let $G(\sigma,\mu)$ is weakly isomorphic to $G'(\sigma',\mu')$ a strong fuzzy graph on one graph does not always have an impact on the other graph.

Example 7 Given a fuzzy graph $G(\sigma,\mu)$ and a strong fuzzy graph $G'(\sigma',\mu')$ as shown in the following figure.

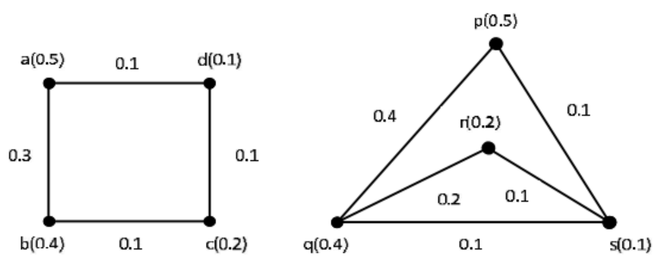

Figure 3. Fuzzy Graph G and G'

with the mapping $h: S \rightarrow S'$ which is a bijective mapping defined by $h(a) = p, h(b) = q, h(c) = r, h(d) = s$. It will be shown that h satisfies the weak isomorphism properties:

1) $\sigma(a) = 0.5$ dan $\sigma'(h(a)) = \sigma(p) = 0.5$
2) $\sigma(b) = 0.4$ dan $\sigma'(h(b)) = \sigma(q) = 0.4$
3) $\sigma(c) = 0.2$ dan $\sigma'(h(c)) = \sigma(r) = 0.2$
4) $\sigma(d) = 0.1$ dan $\sigma'(h(d)) = \sigma(s) = 0.1$
5) $\mu(cd) = 0.1$ dan $\mu'(h(c)h(d)) = \mu'(rs) = 0.1$
6) $\mu(bc) = 0.1$ dan $\mu'(h(b)h(c)) = \mu'(qr) = 0.2$
7) $\mu(ab) = 0.3$ dan $\mu'(h(a)h(b)) = \mu'(pq) = 0.4$
8) $\mu(ad) = 0.1$ dan $\mu'(h(a)h(d)) = \mu'(ps) = 0.1$

So, h is a bijective mapping that satisfies the properties of weak isomorphism such that $G(\sigma, \mu)$ is weakly isomorphic with $G'(\sigma', \mu')$.

It will be shown whether $G(\sigma, \mu)$ is a strong fuzzy graph.

Note that $\mu(bc) = 0.1$ dan $\mu'\big(h(b)h(c)\big) = \mu'(qr) = 0.2$. So $G(\sigma, \mu)$ is not a strong fuzzy graph because there is $0.1 = \mu(bc) \neq \sigma(b) \wedge \sigma(c) = 0.4 \wedge 0.2 = 0.2$.

4. CONCLUSION

Based on the discussion described previously, it is obtained that $G(\sigma, \mu)$ is isomorphic with $G'(\sigma', \mu')$, $G(\sigma, \mu)$ a strong fuzzy graph if and only if $G'(\sigma', \mu')$ is also Fuzzy graphs are strong, but do not apply if $G(\sigma, \mu)$ weak isomorphic with $G'(\sigma', \mu')$. If $G(\sigma, \mu)$ co-weak isomorphic with $G'(\sigma', \mu')$, if $G'(\sigma', \mu')$ is a strong fuzzy graph then $G(\sigma, \mu)$ is also a strong fuzzy graph, but the opposite is not true.

REFERENCES

Gani, A. Nagoor, dan Malarvizhi, J. (2009), Isomorphism Properties on Strong Fuzzy Graphs, *International Journal of Algorithms, Computational and Mathematical, 2* (1).

Gani, A. Nagoor dan J. Malarvizhi. (2010), On Antipodal Fuzzy Graph, *Applied Mathematical Sciences*, 4(43) :2145-2155.

Moderson, John. N dan Premchand S. Nair. (2000), Fuzzy Graphs and Fuzzy Hypergraphs. Physica-Verlag, Heidelberg.

Sameena, K dan M.S Sunitha. (2010), On ss-paths and ss-distance in Fuzzy Graphs, *Journal of Fuzzy mathematics, 5*(1):1-6.

Susilo, Frans. (2006), *Himpunan dan Logika Kabur serta Aplikasinya*. Yogyakarta: Graha Ilmu.

Wilson, J. Robin and John J. Watskin (1990), *Graphs: An Introductory Approach*, New York, University Course Graphs, Network and Design.