

A Historical Analysis of The Padovan Sequence

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ABSTRACT

In the present text, we present some possibilities to formalize the mathematical content and a historical context, referring to a numerical sequence of linear and recurrent form, known as Sequence of Padovan or Cordonnier. Throughout the text some definitions are discussed, the matrix approach and the relation of this sequence with the plastic number. The explicit exploration of the possible paths used to formalize the explored mathematical subject, comes with an epistemological character, still conserving the exploratory intention of these numbers and always taking care of the mathematical rigor approached.

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1. INTRODUCTION.

Aiming at understanding and cause the process of evolution and generalization of mathematical ideas, some of which can be registered through the perception of examples, one can observe this process and conduct discussion around the Padovan numbers. According to Padovan (2002) nature can be considered as something indescribable and unfathomable, and science and art are like a kind of abstraction where we try to understand them.

The Sequence of Padovan, remembered by numbers (1,1,1,2,2,3,4,5,...), by recurrence: $P_n = P_{n-2} + P_{n-3}$ and by the characteristic equation $x^3 - x - 1 = 0$.

Also having historical origins in World War II, and being compared to another recurrent sequence known as Fibonacci sequence, but the latter is of 2nd order.

2. DEFINITIONS AND HISTORICAL CONTEXT

We will start the study of this sequence, defining its recurrence formula and, delaying the initial values of the Padovan sequence in this work. This delay is performed as a way to facilitate the calculations that will be performed during this paper.

Definition 1. The sequence of Padovan or Cordonnier is obtained through the recurrence formula:

$$\begin{cases} P_0 = P_1 = 0, P_2 = 1; \\ P_n = P_{n-2} + P_{n-3}, n \geq 3; \end{cases}$$

Given the above recurrence, we can describe the solution (0,0,1,0,1,1,1,2,2,3,4,5, ...), as a sequence of 3rd order, and

denoted as (P_n) , and being the set that compose it as the numbers of Padovan or Cordonnier. Some other initial values different than the definition of this sequence can be assigned. Hereafter, we will use the terminology Padovan or Cordonnier

sequence and the notation (P_n) representing the same sequence throughout the text.

Thus the sequence of Padovan, which was given by the Italian architect Richard Padovan (1935 -?), Born in the city of Padua (Stewart, 1996), is a kind of relative of a better known one, such as Fibonacci sequence, arithmetic, and integers. Gérard Cordonnier (1907-1977), whose image is remembered in figure 1, has also developed some studies about these numbers, more specifically on the plastic number (radiant number), with that the sequence is also known as the Cordonnier sequence.

Richard Padovan was born in 1935 studied architecture at the Architectural Association, London (1952-57). He believes, however, that his true architecture-oriented education began when he found the work and thought of the Dutch architect Hans van der Laan in 1974. In 1999, he published *Proportion: Science, Philosophy, Architecture*. His latest book, *Towards Universality: Le Corbusier, Mies and De Stijl* in 2002, contrasts the grand philosophical ideals of European modernism. Gérard Cordonnier was born on 7 April 1907 in Bailleul (North), and died suddenly in Lhez, in the High Pyrenees, by road accident, on July 12, 1977.

One can highlight the Dutch Hans Van Der Laan (1904 - 1991) who conducted the architecture course at the Technische

Hogeschool in Delf. He used the primitive Christian basilica of the abbey as an example to train architects in the reconstruction of churches after World War II (VOET; SCHOONJANS, 2012). Laan and his brother sought patterns for architecture through experiments with stones and then with building materials, and eventually discovered a new pattern of measurements where the construction of this occurs through an irrational number, ideal for working on a geometric scale and space objects (rectangles, trapezoids, ellipses, etc.), this number is known as a plastic number or radiant number, and was first studied by Gérard Cordonnier. An analogy is made of the plastic number in relation to music: in music one can play chords, with the radiant number it is possible to compose walls, rooms and the like.



Figure 1. Gerard Cordonnier
Source : Google Imagens

A 2D geometric representation of Padovan (see Figure 2) was developed in Geogebra software to explore the geometry of this sequence. This is composed by the juxtaposition of equilateral triangles respecting a characteristic construction rule. Consider the side 1 triangle highlighted in blue as the initial triangle. The formation of the spiral is given by the addition of a new equilateral triangle to the greater side of the formed polygon, initially the blue triangle. After the addition of the other triangles a new polygon is formed, known as plastic pentagon. The spiral presents itself by connecting two ends of the newly added triangle with an arc.

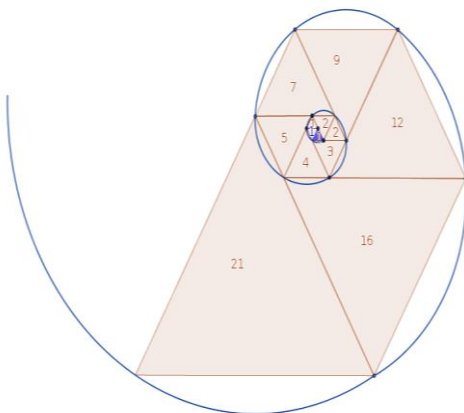


Figure 2. Padovan's spiral
Source : Prepared by the authors

3. RELATIONSHIP WITH PLASTIC NUMBER

As in the Fibonacci sequence, where it is related to the Golden number, a famous mathematical constant used in architecture due to its recurrent presence in symmetrical structures, the Padovan

sequence is also related to a number, called a plastic number (BELINI, 2015).

The characteristic equation of Padovan was obtained considering the relation of recurrence to the right side, obtaining

$$\frac{P_n}{P_{n-2}} = 1 + \frac{P_{n-3}}{P_{n-2}} \text{ soon } \frac{P_n}{P_{n-2}} \cdot \frac{P_{n-1}}{P_{n-2}} = 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}}$$

Assuming that the limit exists, one has to:

$$\lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} = \lim_{n \rightarrow \infty} \psi_n = \psi \text{ for } n \geq 1. \text{ Passing the boundary sign}$$

on equality:

$$\lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} \cdot \frac{P_{n-1}}{P_{n-2}} = \lim_{n \rightarrow \infty} 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}}$$

$$\text{soon } \lim_{n \rightarrow \infty} \psi_n \cdot \psi_{n-1} = \lim_{n \rightarrow \infty} 1 + \frac{1}{\psi_{n-2}} = \psi.$$

Thus, it can be determined that: $\psi \cdot \psi = 1 + \frac{1}{\psi^2}$

determining the following polynomial equation of the Padovan sequence $\psi^3 - \psi - 1 = 0$, that is, $x^3 - x - 1 = 0$ (Sahin, 2017).

The polynomial equation, $\psi^3 - \psi - 1 = 0$ being of third order, has two complex roots and one real root. This can also be verified by defining the locus of the function: $f(\psi) = \psi^3 - \psi - 1$ (see Figure 3).

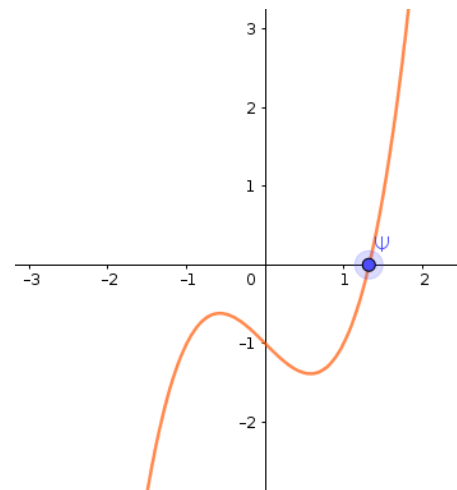


Figure 3. Graph of the Characteristic Equation of Padovan Sequence

Source: Prepared by the Authors

Since this equation presented is not of the complete cubic form, one can use Cardano's formula to find its roots. Consider this equation of type $\psi^3 + p\psi + q = 0$ and its relation to the polynomial of Padovan sequence, we can define the root of a cubic polynomial not complete through the formula:

$$\psi = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

So for $p = q = -1$ you get:

$$\psi = \sqrt[3]{\frac{-(-1)}{2} + \sqrt{\frac{(-1)^2}{4} + \frac{(-1)^3}{27}}} + \sqrt[3]{\frac{-(-1)}{2} - \sqrt{\frac{(-1)^2}{4} + \frac{(-1)^3}{27}}}$$

$$\psi = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx 1.324717957244746...$$

The ratio of convergence ψ is defined as $\psi = \lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} \approx 1.3$ and for Padovan sequence, considering its terms, we must have $\psi = \lim_{n \rightarrow \infty} \frac{P_{18}}{P_{15}} = \frac{37}{28} \approx 1.32$, we can then notice the proximity between the convergence ratio ψ and the root of the Padovan polynomial.

4. MATRIX APPROACH AND PROPERTIES

A way to get any element of a linear and recursive sequence is through the generating matrix Q . This technique was applied to the Fibonacci sequence, and in this subsection the same idea will be applied to the Padovan numbers (Falcon; Plaza, 2007).

The Padovan numbers have a Q matrix of order 3×3 , which when raised to n th power, can get the n th term of this sequence without calculating the recursion. The matrix relationship can be represented by the matrix introduced by Sokhuma (2013) and Seenukul (2015).

Theorem 1. For $n \geq 1$ we have that the generating matrix Q of Padovan sequence having initial values $P_0 = P_1 = 0, P_2 = 1$, is given by :

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Proof. Using the principle of finite induction, we have that $k = n + 1$

$$Q^{n+1} = Q^n \cdot Q^1$$

$$Q^{n+1} = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$Q^{n+1} = \begin{bmatrix} P_n & P_{n-1} + P_n & P_{n+1} \\ P_{n+1} & P_n + P_{n+1} & P_{n+2} \\ P_{n+2} & P_{n+1} + P_{n+2} & P_{n+3} \end{bmatrix} = \begin{bmatrix} P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \\ P_{n+2} & P_{n+4} & P_{n+3} \end{bmatrix}$$

Another five Padovan generating matrices were found by Seenukul (2015), with the same initial values, permuting the rows and columns of this matrix represented in the above theorem, being obtained from the base matrix (Theorem 1) making permutations of the lines and columns, permutation of the latter following the principle used for the line.

Theorem 2. For $n \geq 1$ the generating matrix Q of Padovan sequence is given by:

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+3} & P_{n+2} & P_{n+1} \\ P_{n+1} & P_n & P_{n-1} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof. Using the principle of finite induction performed in Theorem 1, it is possible to demonstrate such a theorem, as well as the other.

This matrix is obtained by performing a permutation as follows: the first line becomes the last line, the second line becomes the first line, and the third line becomes the second. In the same way, the permutation of the columns is performed where the first column becomes the last, the second column goes to the first column and the third goes to the second column, obtaining the matrix of Theorem 2.

Theorem 3. For $n \geq 1$ the generating matrix Q of Padovan sequence is given by:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n+2} & P_n & P_{n+1} \\ P_{n+1} & P_{n-1} & P_n \\ P_{n+3} & P_{n+1} & P_{n+2} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Q^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

In order to obtain this matrix, the base matrix (Theorem 1) was exchanged as follows: the first row was exchanged with the second, and the same columns were exchanged (first with the second column).

Theorem 4. For $n \geq 1$ the generating matrix Q of Padovan is given by:

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n+2} & P_{n+1} & P_{n+3} \\ P_n & P_{n-1} & P_{n+1} \\ P_{n+1} & P_n & P_{n+2} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix obtained from the base matrix of Theorem 1, performing: permutation of the first line becomes the second, permutation of the second line becoming the last and, permutation of the last line becoming the first. The same row permutation procedure was performed for the columns of this matrix.

Theorem 5. For $n \geq 1$ the generating matrix Q of Padovan sequence is given by:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n-1} & P_n & P_{n+1} \\ P_{n+1} & P_{n+2} & P_{n+3} \\ P_n & P_{n+1} & P_{n+2} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This matrix was obtained by performing the permutation procedure of the second line with the third one, fixing the first line. The same process was performed for the permutation of the columns, obtaining the matrix of Theorem 5, from the matrix of Theorem 1.

Theorem 6. For $n \geq 1$ the generating matrix Q of Padovan sequence is given by:

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n+2} & P_{n+3} & P_{n+1} \\ P_{n+1} & P_{n+2} & P_n \\ P_n & P_{n+1} & P_{n-1} \end{bmatrix}$$

so we have:

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Finally, in order to obtain this last valid generating matrix of Padovan sequence, the first line was exchanged with the third one, fixing the second line, also performing the same procedure for the exchange of the columns, from the base matrix of the Theorem 1.

Thereby, some properties related to the Padovan sequence, are shown below:

Property 1. For $n \in \mathbb{N}$, the following property is valid:

$$P_{n+m} = P_n \cdot P_m + P_{n-1} (P_{m-1} + P_{m-2}) + P_{n-2} + P_{m-1}$$

Proof. A property that we can obtain is given through the generating matrix of Theorem 1, performing trivial operations of matrix arithmetic, we have:

$$Q^{n+m} = Q^n \cdot Q^m$$

$$\begin{bmatrix} P_{n+m} & P_{n+m-1} & P_{n+m-2} \\ P_{n+m-1} + P_{n+m-2} & P_{n+m-2} + P_{n+m-3} & P_{n+m-3} + P_{n+m-4} \\ P_{n+m-1} & P_{n+m-2} & P_{n+m-3} \end{bmatrix} = \begin{bmatrix} P_n & P_{n-1} & P_{n-2} \\ P_{n-1} + P_{n-2} & P_{n-2} + P_{n-3} & P_{n-3} + P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \end{bmatrix}$$

$$\begin{bmatrix} P_m & P_{m-1} & P_{m-2} \\ P_{m-1} + P_{m-2} & P_{m-2} + P_{m-3} & P_{m-3} + P_{m-4} \\ P_{m-1} & P_{m-2} & P_{m-3} \end{bmatrix}$$

Providing immediately through the first element of the array, the property:

$$P_{n+m} = P_n \cdot P_m + P_{n-1}(P_{m-1} + P_{m-2}) + P_{n-2} + P_{m-1}$$

Property 2. For $n \in \mathbb{Z}$, we have the property:

$$P_{(n-m)} = (-1)^m P_n \cdot P_m + (-1)^{m+1} \cdot (P_{n-1} \cdot P_{(m+1)} + P_{n-1} \cdot P_{(m+2)} + P_{(m+1)}) + P_{n-2}$$

Proof. Taking property 1, and replacing $P_m = P_{-m}$, we get:

$$P_{n+(-m)} = P_n \cdot P_{(-m)} + P_{n-1}(P_{(-m-1)} + P_{(-m-2)}) + P_{n-2} + P_{(-m-1)}$$

$$P_{(n-m)} = P_n \cdot P_{(-m)} + P_{n-1}(P_{-(m+1)} + P_{-(m+2)}) + P_{n-2} + P_{-(m+1)}$$

$$P_{(n-m)} = (-1)^{m+1} P_n \cdot P_m + (-1)^m P_{n-1}(P_{(m+1)} + P_{(m+2)}) + P_{n-2} + (-1)^m P_{(m+1)}$$

$$P_{(n-m)} = (-1)^m P_n \cdot P_m + (-1)^{m+1} \cdot (P_{n-1} \cdot P_{(m+1)} + P_{n-1} \cdot P_{(m+2)} + P_{(m+1)}) + P_{n-2}$$

5. THE BINET FORMULA

Now we will explore the existence of an explicit formula for calculating the n th term of the sequence, without depending on the recurrence, through the Binet formula. In order to find such a formula, it is necessary to use the characteristic equation of this sequence.

Theorem 7. For $n \geq 0$ the Binet formula of the Padovan sequence is:

$$P(n) = A \cdot x_1^n + B \cdot x_2^n + C \cdot x_3^n$$

where x_1, x_2, x_3 are the roots of the characteristic Padovan equation being one related to the plastic number and the other two complex and conjugated, and:

$$A = \frac{x_3 - x_2}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

$$B = \frac{x_1 - x_3}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

$$C = \frac{x_2 - x_1}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

Proof. Using the initial values as $P_0 = P_1 = 0, P_2 = 1$, we can substitute in the original Binet's formula, denoting the following system:

$$\begin{cases} A + B + C = 0 \\ A \cdot x_1 + B \cdot x_2 + C \cdot x_3 = 0 \\ A \cdot x_1^2 + B \cdot x_2^2 + C \cdot x_3^2 = 1 \end{cases}$$

By solving the linear system, it is possible to obtain:

$$A = \frac{x_3 - x_2}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

$$B = \frac{x_1 - x_3}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

$$C = \frac{x_2 - x_1}{x_1 x_3 (x_1 - x_3) + x_2 x_3 (x_3 - x_2) + x_1 x_2 (x_2 - x_1)}$$

as required.

Yilmaz and Taskara (2013) also calculated the Binet formula from any initialization term.

6. CONCLUSION

This article describes the definition and properties of the Padovan like sequence, in addition to the historical context, its relation to the plastic number, Binet's formula and its matrix representation. These new properties were discovered from properties already known from the Padovan sequence, using mathematical proofs to find the new results. It was also possible to obtain some properties, derived from the Padovan generating matrix, studied in this work.

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The preferred spelling of the word "acknowledgment" in America is without an "e" after the "g". Avoid the stilted expression, "One of us (R. B. G.) thanks . . ." Instead, try "R. B. G. thanks". Put sponsor acknowledgments in the unnumbered footnote on the first page.

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